GENERALIZED LAGRANGE PROBLEMS IN THE CALCULUS OF VARIATIONS*

BY

C. F. ROOS†

I. Introduction

In the new dynamical theory of economics there arises a very general problem which can be said to be a generalization of the Lagrange problem in the calculus of variations.‡ It will not be necessary to consider the formulation of the corresponding economic theory here since I have already done this in another paper.§ It would hardly be fair, however, to introduce the reader to a rather unusual mathematical situation without giving some hint as to its origin. It seems desirable, therefore, to give first a brief economic formulation of the problem whose mathematical aspects will be discussed in this paper.

If there are two producers of an identical commodity C, manufacturing, respectively, amounts $u_1(x)$ and $u_2(x)$ of C per unit time, subject to the respective cost functions $\phi_1(u_1, u_1', u_2, u_2', u_3, u_3', x)$ and $\phi_2(u_1, u_1', u_2, u_2', u_3, u_3', x)$, where $u_3(x)$ is the selling price of C at a time x, then the respective profits obtained during an interval of time $x_0 \leqq x \leqq x_1$ are

$$I_1 = \int_{x_0}^{x_1} [u_3 u_1 - \phi_1(u_1, u_1', u_2, u_2', u_3, u_3', x)]dx,$$

$$I_2 = \int_{x_0}^{x_1} [u_3 u_2 - \phi_2(u_1, u_1', u_2, u_2', u_3, u_3', x)]dx,$$

where ϕ_1 and ϕ_2 are assumed to be continuous with their first and second derivatives with respect to all their arguments, and primes denote derivatives with respect to time x.

The rates of production $u_1(x)$ and $u_2(x)$ and the price $u_3(x)$ will satisfy an equation of demand which in the general case will be of the form

* Presented to the Society, December 31, 1926; received by the editors December 2, 1926.

† National Research Fellow in Mathematics.

‡ For a special example of this problem, see C. F. Roos, *A mathematical theory of competition*, American Journal of Mathematics, vol. 47 (1925), pp. 163–175. See also G. C. Evans, *The dynamics of monopoly*, American Mathematical Monthly, vol. 31 (1921).

§ C. F. Roos, *A dynamical theory of economics*, Journal of Political Economy, vol. 35 (1927). See also Roos, *Dynamical economics*, Proceedings of the National Academy of Sciences, vol. 13 (1927).

$$(1) \qquad G(u_1, u_1', \cdots, u_3', x) = \int_{x_0}^{x} P(u_1, u_1', \cdots, u_3', x, s) ds$$

where G and P have continuity properties similar to those of ϕ_1 and ϕ_2.* Each manufacturer will consider his rate of production to be influenced by the rate of production of his competitor only through the equation of demand, and will desire to determine his own rate of production in such a way that he obtains a maximum profit over some interval of time, say $x_0 \leqq x \leqq x_1$.

The problem of competition for this state of affairs will then be the problem of determining a curve Γ in the space (u_1, u_2, u_3, x), satisfying a functional equation (1), such that an integral I_1, taken along Γ from x_0 to x_1, is a maximum when u_2 is momentarily held fixed, and such that a second integral I_2, also taken along Γ from x_0 to x_1, is a maximum when u_1 is momentarily held fixed. In the usual case the initial time x_0 and the corresponding initial values of the u_i, $i=1, 2, 3$, are fixed. The end time x_1 and the corresponding end values of the u_i may be regarded as fixed or not, depending upon the nature of the problem under consideration. Both cases will be considered at some length in the following paragraphs.

For the particular case $P \equiv 0$ the equation of demand becomes simply a first-order differential equation. For this case the problem of competition can be solved by the methods employed in the classical Lagrange problem in the calculus of variations.† In order to obtain a solution in the classical way we need, however, two sets of Lagrange multipliers, and this makes the problem quite difficult. In the following pages I shall give an analysis for the case in which the rates of production and price are related by a differential equation of demand $G(u_1, u_1', u_2, u_2', u_3, u_3', x) = 0$ without using multipliers, and shall obtain necessary and sufficient conditions. These conditions, although functional in character, seem simpler than the corresponding conditions which would be obtained by the classical analysis.

In discussing the Lagrange problem for several differential equations $G_k(u_1, u_1', \cdots, u_n, u_n', x) = 0$, $k = 1, \cdots, m < n$, I introduce the theory of Volterra integral equations into my analysis to replace the classical theory by means of multipliers. This use of the theory of integral equations enables me to obtain a method for solving the more general problem for which $P_k(u_1, u_1', \cdots, u_n, u_n', x, s) \not\equiv 0$. So far as I know this use of integral equa-

---

* Roos, *Dynamical economics*, loc. cit.

† J. Hadamard, *Leçons sur le Calcul des Variations*, pp. 217 and sequence. See also G. A. Bliss, *The Problem of Lagrange in the Calculus of Variations*, lectures given at the University of Chicago, summer quarter 1925, mimeographed by O. E. Brown, Northwestern University, Evanston, Illinois.

tions is entirely new. As a result the following exposition, although lengthy, does not represent a complete treatment of the subject.

II. Fixed end points. Eulerian equations in functional form

1. Geometrical interpretation of the problem. In order to make our analysis easier to follow let us first examine the problem for which both end points are fixed, and for which $P(u_1, u_1', \cdots, x, s) \equiv 0$, from a geometrical view point. In the hyperspace (u_1, u_2, u_3, x) let $u_2 = u_2(x)$ be any function, continuous with its first derivative, and substitute this value of u_2 in the integrand $F_1(u_1, u_1', \cdots, u_3', x)$ of an integral I_1, corresponding to the I_1 of the introduction, and in the differential equation $G=0$. The function F_1 becomes a function $F_1(u_1, u_1', u_2(x), u_2'(x), u_3, u_3', x)$, and $G=0$ becomes a differential equation $G(u_1, u_1', u_2(x), u_2'(x), u_3, u_3', x) = 0$. The problem of finding $u_1 = y_1(x)$ which maximizes I_1 is thus reduced to the problem of finding a function $y_1(x)$ which maximizes

$$I_1 = \int_{x_0}^{x_1} F_1(u_1, u_1', u_2(x), u_2'(x), u_3, u_3', x)dx,$$

and satisfies $G=0$ and given end conditions whatever they may be.

Again, if $u_1(x) = y_1(x)$ be substituted in the integrand F_2 and in $G=0$, these become, respectively, $F_1(y_1(x), y_1'(x), u_2, u_2', u_3, u_3', x)$ and $G(y_1(x), y_1'(x), u_2, u_2', u_3, u_3', x) = 0$. Choosing the function $u_2(x) = y_2(x)$ so that it satisfies $G=0$ and maximizes

$$I_2 = \int_{x_0}^{x_1} F_2(y_1(x), y_1'(x), u_2, u_2', u_3, u_3', x)dx$$

completes the solution of the problem, for u_1 and u_3 have already been determined in terms of $u_2(x)$. It is important to note that we have assumed the existence of a solution without showing that one actually exists. Conditions for the existence of a solution will be discussed in Part IV of this paper.

2. Admissible arcs and variations. An arc $u_i = u_i(x)$, $i = 1, 2, 3$, which is continuous on the interval $x_0 \leqq x \leqq x_1$, and is such that the interval can be divided into a finite number of subintervals on each of which the functions $u_i(x)$ have continuous derivatives up to and including those of the second order will be called an *admissible arc*. This definition will permit a maximizing arc to have a finite number of corners. All of the elements of an admissible arc shall be required to lie in a simply connected region of a hyperspace (u_1, u_2, u_3, x), and to satisfy the differential equation $G(u_1, u_1', u_2, u_2', u_3.$

$u_3', x)=0$, and, furthermore, to satisfy certain end conditions.* In the following paragraphs all admissible arcs will be regarded as fixed at a fixed x_0, i.e.

$$u_1(x_0) = u_{10}, \quad u_2(x_0) = u_{20}, \quad u_3(x_0) = u_{30}, \tag{2}$$

and either variable or fixed at x_1 depending upon the particular problem under consideration. The behavior of the arcs at x_1 will be pointed out as the work progresses.

If a two-parameter family of admissible arcs $u_i=u_i(x, a, b)$ containing a particular admissible arc Γ for the parametric values $a=b=0$ be given, we shall call the functions

$$\xi_1(x) = \partial u_1(x,0,0)/\partial a, \quad \xi_2(x) = \partial u_2(x, 0, 0)/\partial b$$

partial variations of the family along Γ. Ordinarily we would require a three-parameter family to cover the space (u_1, u_2, u_3, x) completely, but the differential equation $G=0$ and the initial condition $u_3(x_0) = u_{30}$ removes one degree of freedom.

3. The Eulerian equations in functional form. Let us write

$$\begin{aligned} u_\alpha &= y_\alpha + \psi_\alpha(x,a,b) \qquad (\alpha = 1,2), \\ u_3 &= y_3 + \theta(x,a,b), \end{aligned} \tag{3}$$

where the ψ_α are functions, continuous in x, a and b, possessing continuous derivatives of the first order with respect to x, a and b and vanishing when a and b vanish. The functions y_α and y_3 are the functions $u_i(x)$, $i=1, 2, 3$, defining the maximizing curve Γ which we suppose for the present to exist a priori.

In our analysis we shall have to require that the functions F_1, F_2 and G possess continuous derivatives of the second order with respect to each of the arguments u_i, u_i', x, $i=1, 2, 3$, and, furthermore, that $\partial G/\partial u_3' \neq 0$ in the interval $x_0 \leqq x \leqq x_1$. Under these hypotheses the function θ is determined by $G=0$ and the first two equations of (3), except for an arbitrary constant, as a continuous function of x, a, b with continuous derivatives of the first order.

The derivatives $\partial\theta/\partial a$ and $\partial\theta/\partial b$ satisfy the *equations of partial variations*

$$(\partial G/\partial u_1)\partial\psi_1/\partial a + (\partial G/\partial u_1')\partial\psi_1'/\partial a + (\partial G/\partial u_3)\partial\theta/\partial a + (\partial G/\partial u_3')\partial\theta'/\partial a = 0,$$

$$(\partial G/\partial u_2)\partial\psi_2/\partial b + (\partial G/\partial u_2')\partial\psi_2'/\partial b + (\partial G/\partial u_3)\partial\theta/\partial b + (\partial G/\partial u_3')\partial\theta'/\partial b = 0,$$

and will, therefore, also be continuous and have continuous partial derivatives

* See Bliss, loc. cit., p. 3.

of the first order, on account of the continuity requirements on G. We further restrict the ψ_α by the following conditions:

$$\partial\psi_1/\partial a = \xi_1(x), \qquad \partial\psi_1/\partial b = 0,$$
$$\partial\psi_2/\partial a = 0, \qquad \partial\psi_2/\partial b = \xi_2(x),$$

when $a=b=0$. We employ the following notation: $\partial\theta/\partial a=\theta_a(x)$ and $\partial\theta/\partial b=\theta_b(x)$ when $a=b=0$.

Since we have assumed the end values of u_1 and u_2 to be fixed at x_1 as well as at x_0, we can write

$$\xi_1(x_0) = \xi_1(x_1) = \xi_2(x_0) = \xi_2(x_1) = 0.$$

For the parametric values $a=b=0$ the function $\theta(x, 0, 0)=\theta(x)$ must satisfy the differential equations of partial variations

(4A) $\qquad (\partial G/\partial u_1)\xi_1 + (\partial G/\partial u_1')\xi_1' + (\partial G/\partial u_3)\theta_a + (\partial G/\partial u_3')\theta_a' = 0,$

(4B) $\qquad (\partial G/\partial u_2)\xi_2 + (\partial G/\partial u_2')\xi_2' + (\partial G/\partial u_3)\theta_b + (\partial G/\partial u_3')\theta_b' = 0.$

The first of these determines θ_a in terms of ξ_1 and the partial derivatives of G with respect to u_1 and u_1', except for a constant, whereas the second determines θ_b in terms of ξ_2 and the partial derivatives of G with respect to u_2 and u_2', except for a constant. Choosing these constants so that each of the partial variations $\theta_a(x_0)$ and $\theta_b(x_0)$ vanishes implies that the total variation of the function θ be zero at x_0, i.e. $\delta\theta=\theta_a\delta a+\theta_b\delta b=0$ at $x=x_0$. Conversely, since the ψ_α are arbitrary, the vanishing of $\delta\theta$ implies the vanishing of both θ_a and θ_b. The equations (4) and the initial conditions (2), therefore, completely determine the variations of u_3. The functions u_1 and u_2 have thus been classified as independent functions in a manner similar to the way in which variables are classified in the ordinary theory of maxima and minima of functions.

If functions $u_i(x, a, b)$, defining a two-parameter family of admissible arcs containing Γ for the parametric values $a=b=0$, are substituted in I_1. this integral becomes a function of a and b defined by

$$I_1(a,b) = \int_{x_0}^{x_1} F_1(u_1(x,a,b), u_1'(x,a,b), \cdots, u_3'(x,a,b), x)dx.$$

The partial variation of this integral with respect to a reduces to

$$(\partial I_1/\partial a)\delta a = \int_{x_0}^{x_1} [(\partial F_1/\partial y_1)\xi_1 + (\partial F_1/\partial y_1')\xi_1' + (\partial F_1/\partial y_3)\theta_a + (\partial F_1/\partial y_3')\theta_a']dx\delta a$$

for $a=b=0$.

Instead of proceeding in the classical way we shall solve the differential equation (4) for θ_a and develop a theory without the use of Lagrange multipliers.* This procedure seems to be more directly an extension of the ordinary theory of maxima and minima; it allows us to obtain the Weierstrass, Legendre and Jacobi conditions by an analysis which is simpler than that used in the classical theory, and, furthermore, it leads to a method for solving the Lagrange problem when the differential equations are replaced by functional equations of the type (1). We proceed as follows:

Since by hypothesis $\partial G/\partial y_3'$ is not zero in the interval $x_0 \leqq x \leqq x_1$, the solution of (4) for θ_a is

$$\theta_a = \int_{x_0}^{x} e^{V_t}[(\partial G_3'/\partial y_1)\xi_1 + (\partial G_3'/\partial y_1')\xi_1']dt, \tag{5}$$

where the following notation has been introduced: $(\partial G/\partial y_3)/(\partial G/\partial y_3') = -\partial G_3'/\partial y_3$; $(\partial G/\partial y_1)/(\partial G/\partial y_3') = -\partial G_3'/\partial y_1$; $(\partial G/\partial y_1')/(\partial G/\partial y_3') = -\partial G_3'/\partial y_1'$; $V_t = \int_t^x (\partial G_3'/\partial y_3)ds$. The assumption $\theta_a(x_0) = 0$, made above, does not necessarily impose a limitation on this method, for, if u_3 were variable at x_0, the solution for θ_a would be the solution above plus the variation of u_3 at x_0.

Differentiation of (5) with respect to x determines θ_a' by the formula

$$\begin{aligned}\theta_a' = {} & (\partial G_3'/\partial y_1)\xi_1 + (\partial G_3'/\partial y_1')\xi_1' \\ & + (\partial G_3'/\partial y_3)\int_{x_0}^{x} e^{V_t}[(\partial G_3'/\partial y_1)\xi_1 + (\partial G_3'/\partial y_1')\xi_1']dt.\end{aligned} \tag{6}$$

When the values of θ_a and θ_a' as given by (5) and (6) are substituted in the expression defining the partial variation of I_1 with respect to a, it becomes for $a = b = 0$

$$\begin{aligned}(\partial I_1/\partial a)\delta a = \int_{x_0}^{x_1}\Big[& [\partial F_1/\partial y_1 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1]\xi_1 + [(\partial F_1/\partial y_3')\partial G_3'/\partial y_1' \\ & + \partial F_1/\partial y_1']\xi_1' + [\partial F_1/\partial y_3 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_3]\int_{x_0}^{x} e^{V_t}[(\partial G_3'/\partial y_1)\xi_1 \\ & + (\partial G_3'/\partial y_1')\xi_1']dt\Big]dx.\end{aligned}$$

An application of Dirichlet's formula for changing the order of integration of an iterated integral, followed by an interchange of t and x, the parameters of integration, yields the equation

* Hadamard, loc. cit., Chapter VI, gives the classical theory.

$$(\partial I_1/\partial a)\delta a = \int_{x_0}^{x_1}\Big[[\partial F_1/\partial y_1 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1 + (\partial G_3'/\partial y_1)W_1]\xi_1$$
$$+ [\partial F_1/\partial y_1' + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1' + (\partial G_3'/\partial y_1')W_1]\xi_1'\Big]dx,$$

where

$$W_1 = \int_x^{x_1} e^{V_3}[\partial F_1/\partial y_3 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_3]dt.$$

Since $\xi_1(x)$ vanishes at x_0 and x_1 by hypothesis, an integration by parts performed on the terms involving $\xi_1(x)$ of the partial variation of I_1 with respect to a furnishes the expression

$$(\partial I_1/\partial a)\delta a = \int_{x_0}^{x_1}\Big[-\int_{x_0}^{x}[\partial F_1/\partial y_1 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1 + (\partial G_3'/\partial y_1)W_1]dt$$
$$+ [\partial F_1/\partial y_1' + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1' + (\partial G_3'/\partial y_1')W_1]\Big]\xi_1'(x)dx,$$

where the coefficient of $\xi_1'(x)$ is continuous because of the continuity requirements on F_1 and G.

If I_1 is to be a maximum along the curve Γ, it is necessary that $(\partial I_1/\partial a)\delta a$ be zero for all values of the functions $\xi_1(x)$. By a well known theorem of the calculus of variations it follows that the coefficient of $\xi_1'(x)$ must be a constant, that is,

$$\partial F_1/\partial y_1' + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1' + (\partial G_3'/\partial y_1')W_1$$
$$= \int_{x_0}^{x}[\partial F_1/\partial y_1 + (\partial F_1/\partial y_3')\partial G_3'/\partial y_1 + (\partial G_3'/\partial y_1)W_1]dt + C_1, \tag{7}$$

where C_1 is a constant to be determined by the initial conditions.

An entirely similar analysis applied to I_2 yields the necessary condition

$$\partial F_2/\partial y_2' + (\partial F_2/\partial y_3')\partial G_3'/\partial y_2' + (\partial G_3'/\partial y_2')W_2$$
$$= \int_{x_0}^{x}[\partial F_2/\partial y_2 + (\partial F_2/\partial y_3')\partial G_3'/\partial y_2 + (\partial G_3'/\partial y_2)W_2]dt + C_2. \tag{8}$$

The functional-differential equations (7) and (8) are the analogues of the Euler equations in the Du Bois-Reymond form.* Wherever the maximizing curve Γ has a continuously turning tangent we can differentiate (7) and (8) with respect to x and obtain functional-differential equations which involve

* Du Bois-Reymond, Mathematische Annalen, vol. 15 (1879), p. 313.

second-order derivatives and which are the analogues of the Euler equations. We can, therefore, state the following theorem.

THEOREM 1. *In order that an admissible arc* Γ *in the space* (u_1, u_2, u_3, x), *satisfying a differential equation* $G(u_1, u_1', u_2, u_2', u_3, u_3', x)=0$ *and initial conditions* $u_i(x_0)=u_{i0}$, $u_i(x_1)=u_{i1}$, *maximize an integral* I_1 *when* u_2 *is not allowed to vary and at the same time maximize a second integral* I_2 *when* u_1 *is not allowed to vary, it is necessary that this curve satisfy the functional-differential equations* (7) *and* (8). *If the maximizing curve has a continuously turning tangent at* x, $x_0 \leqq x \leqq x_1$, *it must satisfy the equations*

$$\begin{aligned}(9)\qquad \partial F_k/\partial y_k + (\partial F_k/\partial y_3')\partial G_3'/\partial y_k + (\partial G_3'/\partial y_k)W_k - \frac{d}{dx}[\partial F_k/\partial y_k' \\ + (\partial F_k/\partial y_3')\partial G_3'/\partial y_k' + (\partial G_3'/\partial y_k')W_k] = 0 \quad (k = 1,2),\end{aligned}$$

obtained by differentiating (7) *and* (8) *with respect to* x.

Functional-differential equations of the type (9), with our form of W_k, have not been discussed in the literature. It would be desirable to be able to say that a unique solution of these equations plus the differential equation $G=0$ exists whenever end values $u_i(x_0)=u_{i0}$ and $u_i(x_1)=u_{i1}$ are given. This problem will not be discussed in the present paper.* It may be mentioned, however, that I have already exhibited a special example for which the system (9) reduces to a system of Volterra integral equations, and have actually found the solution.† Let us examine (7) and (8) from a different point of view.

In particular if $F_1 \equiv F_2$, the problem reduces to a strict Lagrange problem. No assumption which would prevent this has been made, hence we have the following

COROLLARY. *The equations resulting from* (7) *and* (8) *by putting* $F_1 \equiv F_2$ *must be satisfied by a curve satisfying a differential equation* $G=0$ *and initial conditions* $u_i(x_0)=u_{i0}$, $u_i(x_1)=u_{i1}$ *if this curve is to maximize an integral*

$$I = \int_{x_0}^{x_1} F_1(u_1, u_1', u_2, u_2', u_3, u_3', x)dx$$

in which both u_1 *and* u_2 *vary independently.*

---

* L. M. Graves, *Implicit functions and differential equations in general analysis*, these Transactions, vol. 29, pp. 515–552, gives imbedding and existence theorems for a system which includes (9) as a special case. If y_k''' is continuous, we can reduce (9) to a differential equation of the third order by a differentiation, because of the form of W_k, and existence theorems for differential equations will apply.

† Roos, *A mathematical theory of competition*, loc. cit., p. 167.

The methods of this part can be extended without difficulty to the case for which there are n integrals

$$I_h = \int_{x_0}^{x_1} F_h(u_1, u_1', \cdots, u_n, u_n', u_{n+1}, u_{n+1}', x)dx \qquad (h = 1, 2, \cdots, n)$$

and one differential equation $G(u_1, u_1', \cdots, u_n, u_n', u_{n+1}, u_{n+1}', x) = 0$, in which case n functional equations of the type (7) result.

III. Variable end points. Analogues of Weierstrass and Legendre conditions

4. Problem with one end point variable. In the preceding paragraphs a problem in simultaneous maxima for fixed end points has been considered. The problem is even more interesting when one end parameter, say x_1, and the corresponding end values are allowed to vary.

Consider the problem of determining a curve Γ in the space (u_1, u_2, u_3, u_4, x) satisfying a differential equation

$$G(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x) = 0$$

such that an integral

$$I_1 = \int_{x_0}^{x_1} F(u_1, u_1', \cdots, u_4, u_4', x)dx$$

is a maximum when u_1 and u_2 are allowed to vary independently, but not u_3, and such that a second integral

$$I_2 = \int_{x_0}^{x_1} F_2(u_1, u_1', \cdots, u_4, u_4', x)dx$$

is a maximum when u_3 is allowed to vary independently, but not u_1 and u_2. We assume the end parameter x_0 and the end values $u_i(x_0) = u_{i0}$ to be fixed, and the end parameter x_1 and the corresponding end values of the u_i to be variable. Let us assume as we did in Part II that $\partial G/\partial u_4' \neq 0$ for the region which contains admissible arcs $u_i(x)$, $i = 1, 2, 3, 4$, and that the functions F_α, $\alpha = 1, 2$, and G are continuous in $u_1, u_2, u_3, u_4, u_1', u_2', u_3', u_4', x$ and have continuous partial derivatives of the first order with respect to these arguments.

5. Functional transversality conditions. In the functions F_α and G replace the functions $u_i(x)$, $i = 1, 2, 3, 4$, by a set $u_i = f_i(x, a_1, a_2, a_3)$, where the f_i are functions of x and parameters a_1, a_2 and a_3, continuous and admitting continuous derivatives up to the second order with respect to x and these parameters in the domain $0 \leqq a_1 \leqq h$; $0 \leqq a_2 \leqq h$; $0 \leqq a_3 \leqq h$; $x_0 \leqq x \leqq x_1$. The functions f_1, f_2 and f_3 are otherwise arbitrary, but f_4 is determined by $G=0$ and the initial condition $u_4(x_0)=u_{40}$. Let the limit of integration x_1 be a similar function of the parameters a_σ, $\sigma=1, 2, 3$, i.e. $x_1=\phi(a_1, a_2, a_3)$.

By the ordinary rules of differentiation the differential of the integral I_1, which is also a function of the a_σ, is for u_3 (= constant)

$$dI_1 = [F_1(u_1,u_1', \cdots, u_4,u_4',x)\delta x]^{x_1} + \int_{x_0}^{x_1}[(\partial F_1/\partial u_i)\delta f_i + (\partial F_1/\partial u_i')\delta f_i']dx, \tag{10}$$

where $\delta f_i = (\partial F_i/\partial a_1)\delta a_1 + (\partial f_i/\partial a_2)\delta a_2 + (\partial f_i/\partial a_3)\delta a_3$ and i is an umbral index for the values 1, 2, 4, but not for 3, according to the convention that whenever a literal suffix appears twice in a term that term is to be summed for values of the suffix.* The variation of u_3 in F_1 is by hypothesis equal to zero, hence $\delta f_3 \equiv 0$.

As already stated the variations of f_1 and f_2 are to be arbitrary (except for continuity properties), but we can not take the variation of f_4 to be arbitrary, for it is determined by the differential equation of partial variation

$$(\partial G/\partial u_4)\delta f_4 + (\partial G/\partial u_4')\delta f_4' + (\partial G/\partial u_k)\delta f_k + (\partial G/\partial u_k')\delta f_k' = 0,$$

where k is an umbral index taking on the values 1 and 2 only. Since $\partial G/\partial u_4'$ does not vanish and is continuous in the interval by hypothesis, and, furthermore, since $\delta(df/dx) = (d/dx)\delta f$, this expression can be regarded as a first-order differential equation for the determination of δf_4 in terms of δf_1 and δf_2 and the initial value of δf_4 at $x=x_0$. Since we have supposed $\delta f_4(x_0)=0$, we may write

$$\delta f_4 = \int_{x_0}^{x} e^{V_4}[(\partial G_4'/\partial u_k)\delta f_k + (\partial G_4'/\partial u_k')\delta f_k']dt,$$

where the expressions of the form $\partial G_4'/\partial u_k$, etc. have meanings similar to the corresponding ratios defined in (5). As in (5) we determine the value of $\delta f_4'$ by differentiation of the above expression. If the values of δf_4 and $\delta f_4'$ so found be substituted in (10), it becomes

$$F_1\delta x]^{x_1} + \int_{x_0}^{x_1}\Big[[\partial F_1/\partial u_k + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k)]\delta f_k$$
$$+ [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k')]\delta f_k'$$
$$+ [\partial F_1/\partial u_4 + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_4)]\int_{x_0}^{x} e^{V_4}[(\partial G_4'/\partial u_k)\delta f_k + (\partial G_4'/\partial u_k')\delta f_k']dt\Big]dx.$$

---

* See A. S. Eddington, *The Mathematical Theory of Relativity*, p. 50.

An application of Dirichlet's formula to the iterated integral followed by an interchange of the parameters x and t as before reduces the above formula for δI_1 to

$$(10B)\quad \delta I_1 = F_1\delta x]^{x_1} + \int_{x_0}^{x_1}\Big[[\partial F_1/\partial u_k + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k) + (\partial G_4'/\partial u_k)W_1]\delta f_k + [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1]\delta f_k'\Big]dx,$$

where

$$W_1 = \int_{x_1}^{x} e^{V_4}[(\partial F_1/\partial u_4')(\partial G_4'/\partial u_4) + \partial F_1/\partial u_4]dt.$$

Since the $f_k(x, a_1, a_2, a_3)$, $k=1, 2$, have by hypothesis continuous second derivatives with respect to x, the formula for integration by parts can be applied to the second member of (10B), so that*

$$\delta I_1 = \Big[F_1\delta x + [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1]\delta f_k\Big]_{x_0}^{x_1} + \int_{x_0}^{x_1}\Big[\partial F_1/\partial u_k + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k) + (\partial G_4'/\partial u_k)W_1 - \frac{d}{dx}[\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1]\Big]\delta f_k dx,$$

where k is umbral as before.

By definition $\delta f_k = (\partial F_k/\partial a_i)\delta a_i$, where i is umbral with range 1, 2, 3, hence the variation of u_k is given by $\delta u_k = u_k'\delta x + \delta f_k$. If the value of δf_k defined by this equation be substituted in δI_1, the following formula results:

$$\delta I_1 = \Big[\partial F_1/\partial u_k' + [(\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1][\delta u_k - u_k'\delta x]\Big]_{x_0}^{x_1} + \int_{x_0}^{x_1}\Big[\partial F_1/\partial u_k + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k) + (\partial G_4'/\partial u_k)W_1 - \frac{d}{dx}[\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1]\Big]\delta f_k dx.$$

We define an arc $u_i = u_i(x)$, $i=1, 2, 3, 4$, as an *extremal arc* if it has continuous derivatives du_i/dx and d^2u_i/dx^2 in the interval $x_0 \leqq x \leqq x_1$, and if, furthermore, it satisfies the differential equation $G(u_1, u_1', \cdots, u_4, u_4', x) = 0$, the set of two equations

* Hadamard, loc cit., p. 60.

$$(11)\qquad \partial F_1/\partial u_k + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k) + (\partial G_4'/\partial u_k)W_1 - \frac{d}{dx}[\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k') + (\partial G_4'/\partial u_k')W_1] = 0 \qquad (k = 1,2),$$

and a similar set for the integral I_2.

If an extremal Γ is to maximize I_1 for u_3 constant, it is necessary that the differential

$$\delta I_1(\Gamma) = F_1\delta x_1 + [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')\partial G_4'/\partial u_k'][\delta u_k(x_1) - u_k'\delta x_1]$$

vanish for all possible choices of δx_1 and $\delta u_k(x_1)$, $k=1, 2$. We can thus state the transversality theorem:

THEOREM 2. *If for an admissible arc* Γ, *one of whose end points is fixed at* x_0 *while the other varies over a* V_3 *defined by* $u_3=$*constant and a differential equation* $G=0$, *the value* $I_1(\Gamma)$, *for* $u_3=$*constant*, $G=0$, *is a maximum with respect to the values of* I_1 *on neighboring admissible arcs, issuing from the same fixed point* 0, *then at the intersection point* 1 *of* Γ *with* V_3, *the directional coefficients of* V_3 *and the element* $(u_1, u_1', \cdots, u_4', x)$ *of* Γ *must satisfy the relations*

$$(12)\qquad \begin{aligned} &F_1(u_1,u_1', \cdots, u_4', x) - [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k')]u_k' = 0,\\ &\partial F_1/\partial u_i' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_i') = 0^* \qquad (i = 1,2).\end{aligned}$$

If we apply a similar analysis to the integral I_2, for u_1 and u_2 constant, we obtain a differential

$$\delta I_2(\Gamma) = F_2\delta x_1 + [\partial F_2/\partial u_3' + (\partial F_2/\partial u_3')(\partial G_4'/\partial u_3')][\delta u_3(x_1) - u_3'(x_1)\delta x_1]$$

along an extremal for the integral I_2. If, therefore, Γ is also to maximize I_2 for u_1 and u_2 constant, then at 1, the intersection of Γ with V_2, defined by $u_1=$constant, $u_2=$constant and $G=0$, it is necessary that the equations

$$(13)\qquad \begin{aligned} &F_2(u_1,u_1', \cdots, u_4', x) + [\partial F_2/\partial u_3' + (\partial F_2/\partial u_4')(\partial G_4'/\partial u_3')]u_3' = 0,\\ &\partial F_2/\partial u_3' + (\partial F_2/\partial u_4')(\partial G_4'/\partial u_3') = 0\end{aligned}$$

* From a consideration of the classical theory of the Lagrange problem with second end point variable we would expect to have four transversality conditions instead of three as given by (12), but we have not used the condition that $\delta u_4(x_1)$ is arbitrary, since it is a function of arbitrary functions δf_k, and hence we lack this condition. If we perform an integration by parts on the term in $\delta f_i'$ of (10), and then substitute for δf_i as we did above, we obtain a term $(\partial F_1/\partial u_4')\delta f_4$ besides terms in δf_k. Since δf_4 is arbitrary at x_1 it follows that $\partial F_1/\partial u_4'=0$ at $x=x_1$. We may, therefore, by the help of (12) write the transversality condition as

$$(12A)\qquad \begin{aligned} &F_1(u_1, u_1', \cdots, u_4', x) - (\partial F_1/\partial u_k')u_k' = 0,\\ &\partial F_1/\partial u_k' = 0 \qquad (k = 1, 2, 4).\end{aligned}$$

The equations (12A) are the analogues of the usual transversality conditions. (See Bliss, loc. cit., p. 167.)

hold. In equations (12) and (13) we have five equations for the determination of the four end values x_1, $u_h(x_1)$, $h=1, 2, 3$, of an extremal Γ. In general, therefore, the problem of simultaneous maxima is not possible for the case for which the second end parameter x_1 is required to be the same for both I_1 and I_2. Hence, it will be understood in our work that x_1 has, in general, different values for I_1 and I_2. The conditions (12) and (13) are functional in form and will be called *functional transversality conditions*.

6. Analogue of the Weierstrass necessary condition. By the aid of the expression for $\delta I_1(\Gamma)$ we can state the following theorem:

THEOREM 3. *The value of an integral I_1, taken along a two-parameter family of extremal arcs E_{01} determined by the equations $u_k=f_k(x, a_1, a_2)$, $k=1, 2, 4$, $G=0$, and the hypersurface $u_3=f_3(x)=constant$, one of whose end points, x_0, is fixed while the other, x_1, varies, has a differential*

$$dI_1 = F_1(u_1, p_1, u_2, p_2, u_3, u_3', u_4, p_4, x)dx_1 + [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k')][du_k(x_1) - p_k dx_1],$$

where at the point 1, the differentials dx_1 and du_k are those belonging to V_3 ($u_3=constant$) described by the end points of the extremals, while the u_i, u_3, p_i and u_3' refer to the extremal E_{01}. The functions F_1 and G have arguments (u_1, p_1, u_2, p_2, u_3, u_3', u_4, p_4, x), where the p_1, p_2 and p_4 are the directional coefficients of the extremal E_{01} for $u_3=constant$.

There is an entirely analogous theorem for the integral I_2. For I_2 the functions F_2 and G have arguments (u_1, u_1', u_2, u_2', u_3, p_3, u_4, p_4, x).

The integral of dI_1 corresponds to the Hilbert integral and possesses similar properties. In a manner analogous to the classical method of the calculus of variations it is possible to obtain the necessary conditions of Weierstrass and Legendre and to obtain sufficient conditions for relative strong and weak maxima.* To do this we first define an extremal field in the sense in which we shall use it in this chapter.

We shall say that a connected region R of the space (u_1, u_2, u_3, u_4, x) is a simply covered *extremal field* if there exists a family of extremals dependent upon three parameters such that one and only one extremal of this simply covered field passes through every point of R, and if, furthermore, the directional coefficients $du_h/dx=p_h(u_1, \cdots, u_4, x)$, $h=1, \cdots, 4$, of the tangent to the extremal, which passes through the point (u_1, u_2, u_3, u_4, x), are continuous functions, admitting continuous partial derivatives in R

* For the classical analysis see Hadamard, loc. cit., p. 364. See also Bliss, loc. cit., p. 50.

up to the second order. We shall assume that such a field exists and that it contains V_3.

It is quite evident that along an extremal arc of a field, the integral $I_1^* = \int dI_1$ has the same value as I_1, for $\delta u_i = p_i \delta x$ along an extremal, and the integrand of I_1^* thus reduces to the integrand of I_1.

To obtain an analogue of the Weierstrass condition we select a point (3) on E_{01}, the extremal which we are assuming to give the desired maximum, and through this point (3), holding $f_3(x, a_1, a_2, a_3) = u_3(x)$ constant, pass an otherwise arbitrary curve C_{12} with continuously turning tangent in R. We note that R may be partly bounded by V_3, so that when (3) is at (1) the curves C_{12} are further limited. Such a curve C_{12} will have equations $u_1 = U_1(t)$, $u_2 = U_2(t)$, $u_3 = u_3(t)$, $u_4 = U_4(t)$.

We join the fixed point 0 to a movable point 2 on C_{12} by a one-parameter family of arcs E_{02}, containing E_{01} as a member when the point 2 is in the position 1. We choose the parameter t on C_{21} increasing as 2 moves towards 1, noting that the arc length s is a possible t. If E_{01} is to give a maximum for admissible arcs in R, i.e., $I_1(E_{02}+C_{21}) \leqq I_1(E_{01})$, where C_{12} and E_{02} are obtained by putting $a_3 = 0$, it follows that $dI_1(E_{02}+C_{21}) = dI_1(E_{02}-C_{12})$ must be $\geqq 0$ for 2 sufficiently close to 1, and in particular at 1 itself, that is, $dI_1(C_{12}-E_{02}) \leqq 0$ must hold.

This differential is given by the value at the point 1 of the expression

$$\begin{aligned} &F_1(U_1, U_1', U_2, U_2', u_3, u_3', U_4, U_4')\delta x \\ &\qquad - F_1(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)\delta x \\ &\qquad - [\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u_k')][\delta u_k - u_k'\delta x], \end{aligned}$$

the differentials in this expression belonging to the arc C_{12} and, therefore, satisfying the equation $\delta u_k = U_k' \delta x$. At the point 1 the coördinates of C_{12} and E_{02} are equal, so that this expression can be written as

$$\begin{aligned} dI_1(C_{12} - E_{02}) = {} & [F_1(u_1, U_1', u_2, U_2', u_3, u_3', u_4, U_4', x) \\ & - F_1(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x) \\ & - [U_k' - u_k'][\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial G_4'/\partial u')]]\delta x. \end{aligned}$$

We shall call the coefficient of δx in the above expression E_1, because it is an analogue of the Weierstrass E-function.† Since the differential $dI_1(C_{12}-E_{01})$ must be negative or zero for an arbitrarily selected point 1 and an arc C through it, we have the following theorem:

† Bliss, loc. cit., p. 130.

THEOREM 4. *At every element* $(u_1, u_1', \cdots, u_4, u_4', x)$ *of an arc* E_{01} *which maximizes an integral* I_1 *when* u_3 *is not allowed to vary and which satisfies a differential equation* $G(u_1, u_1', \cdots, u_4, u_4', x)=0$, *the Weierstrass condition*

$$E_1(u_1,u_1',U_1',u_2,u_2',U_2',u_3,u_3',u_4,u_4',U_4',x) \leqq 0 \tag{14}$$

must be satisfied for every admissible set $(u_1', U_1, u_2, U_2', u_3, u_3', u_4, U_4')$, *different from* $(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)$, *for all values of the coördinates* (u_1, u_2, u_3, u_4, x) *in the region* R.

A similar analysis applied to the integral I_2 yields the following theorem:

THEOREM 5. *At every element* $(u_1, u_1', \cdots, u_4, u_4', x)$ *of an arc* E_{01} *which maximizes an integral* I_2 *when* u_1 *and* u_2 *are not allowed to vary and which satisfies a differential equation*, $G=0$, *the condition*

$$E_2(u_1,u_1',u_2,u_2',u_3,u_3',U_3',u_4',u_4',U_4',x) \leqq 0 \tag{15}$$

must be satisfied for every admissible set $(u_1, u_1', u_2, u_2', u_3, U_3', u_4, U_4', x)$ *different from* $(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)$, *for all values of the coördinates* (u_1, u_2, u_3, u_4, x) *in the region* R.

The conditions (11), (12), (13), (14) and (15) are necessary conditions which must be satisfied by an arc E_{01} which furnishes a solution of the problem of this paper. In the following paragraph another necessary condition will be obtained.

7. Analogue of the Legendre necessary condition. For brevity we will consider only the first integral I_1. If the function $F_1(u_1, U_1', u_2, U_2', u_3, u_3', u_4, U_4', x)$ be expanded by means of Taylor's formula, the following expression is obtained:

$$\begin{aligned} F_1(u_1,U_1',u_2,U_2',u_3,u_3',u_4,U_4',x) &= F_1(u_1,U_1',u_2,U_2',u_3,u_3',u_4,U_4',x) \\ &\quad + [U_k' - u_k'][\partial F_1/\partial u_k' + (\partial F_1/\partial u_4')(\partial u_4'/\partial u_k')] \\ &\quad + \tfrac{1}{2}[U_k' - u_k'][U_h' - u_h']\partial A_{1h}/\partial u_k', \end{aligned}$$

where $A_{1h}=\partial F_1/\partial u_h'+(\partial F_1/\partial u_4')(\partial u_4'/\partial u_h')$ and h and k are umbral indices with range 1, 2. The arguments of A_{1h} are $(u_1, u_1'+\theta(U_1'-u_1'), u_2, u_2'+\theta(U_2'-u_2'), u_3, u_3', u_4, u_4'+\theta(U_4'-u_4'), x)$, where $0<\theta<1$. It should be noted that in this formula u_3 is not allowed to vary.

Since the partial derivative of G with respect to u_k determines $\partial u_4'/\partial u_k'$, the function E_1 is given by the formula

$$E_1(u_1,u_1',U_1',\cdots,u_4,u_4',U_4',x) = \tfrac{1}{2}(U_k' - u_k')(U_h' - u_h')(\partial A_{1h}/\partial u_k').$$

Let us write $T=U_1'-u_1'$, $W=U_2'-u_2'$ and $V=U_3'-u_3'$. We may then by the help of (14) state the following theorem:

THEOREM 6. *If the extremal E_{01} makes I_1 a maximum when u_3 is not allowed to vary, and at the same time makes I_2 a maximum when u_1 and u_2 are not allowed to vary, it is necessary that the quadratic differential forms*

$$T^2[\partial A_{11}/\partial u_1'] + TW[\partial A_{12}/\partial u_1' + \partial A_{11}/\partial u_2'] + W^2[\partial A_{12}/\partial u_2'], \tag{16}$$

$$V^2[\partial A_{23}/\partial u_3'] \tag{17}$$

be definite negative forms for all systems of finite values of u_1', u_2', u_3' and u_4', when the point (u_1, u_2, u_3, u_4, x) remains in the domain R.†

In (16) u_3 is not allowed to vary and in (17) u_1 and u_2 are not allowed to vary. In particular if we let U_k' approach u_k' in (16) we obtain a condition analogous to the Legendre condition.‡

8. **The analogue of the Jacobi condition.** We proceed now to determine an analogue of the Jacobi condition for the problem of simultaneous maxima. Let E_{02} and E_{03} be two extremals of a two-parameter family, $u_3 = \text{constant}$, $u_4(x_0) = u_{40}$, through the point 0, and suppose that these extremals touch an envelope N of the family at their end points 2 and 3. Since the differential dI_1 of §6 is a total differential, u_3 being constant, the integral $I_1^* = \int dI_1$ around a closed contour C is zero. As already pointed out in §6, I_1^* along an extremal is identically equal to I_1, hence

$$I_1(E_{03}) - I_1(E_{02}) = I_1^*(N_{23}).$$

The differentials dx, du_i, $i = 1, \cdots, 4$, at a point of the envelope satisfy the equations $du_i = p_i dx$ with the slope p of the extremal tangent to N at that point. It follows then that $I_1^*(N_{23})$ is the same as $I_1(N_{23})$; hence the following theorem:

THEOREM 7. *If E_{02} and E_{03} are two members of a two-parameter family, $u_3 = \text{constant}$, of extremals through the fixed point* 0, *and if these touch an envelope N of the family at their end points* 2 *and* 3, *then the values of the integral I_1 along the arcs E_{02}, E_{03} and N_{32} satisfy the equation $I_1(E_{02}) + I_1(N_{32}) = I_1(E_{03})$ for every position of the point* 3 *preceding* 2 *on N.*

This theorem is the analogue of the envelope theorem of the calculus of variations.§ We have also a similar theorem for the integral I_2, for u_1 and u_2 held constant.

---

† For definition of definite negative forms see M. Bôcher, *Introduction to Higher Algebra*, p. 150.

‡ Hadamard, loc. cit., p. 391.

§ Bliss, loc. cit., pp. 140–141.

Consider now the value of $I_1'(0)=\delta I_1(0)$ given by equation (10B) for all end values fixed. Let $\lambda_k(u_1, u_1', \cdots, u_4, u_4', x)$ and $\zeta_k(u_1, u_1', \cdots, u_4, u_4', x)$ be, respectively, the coefficients of δf_k and $\delta f_k'$ in this expression. In order to simplify the notation further let $\delta f_k = \xi_k$. We can then write (10B) as

$$I_1'(0) = \int_{x_0}^{x_1} [\lambda_k \xi_k + \zeta_k \xi_k']dx,$$

where k is umbral and ranges over 1 and 2 only. By a differentiation of this expression we obtain

$$\begin{aligned} I_1''(0) = \int_{x_0}^{x_1} [(\partial\lambda_k/\partial u_h)\xi_k\xi_h &+ (\partial\lambda_k/\partial u_h')\xi_h'\xi_k + (\partial\lambda_k/\partial u_4)\xi_k\delta u_4 \\ &+ (\partial\lambda_k/\partial u_4')\xi_k\delta u_4' + (\partial\zeta_k/\partial u_h)\xi_h\xi_k' + (\partial\zeta_k/\partial u_h')\xi_k'\xi_h' \\ &+ (\partial\zeta_k/\partial u_4)\xi_k'\delta u_4 + (\partial\zeta_k/\partial u_4')\xi_k'\delta u_4']dx, \end{aligned}$$

since ξ_3 is zero by hypothesis. The variations δu_4 and $\delta u_4'$ are to be determined by the equations of partial variation of §5. We could carry out the substitution and application of Dirichlet's formula as before, and obtain an equation analogous to Jacobi's differential equation.* As one can readily see, this equation would be functional-differential in form, and would, therefore, be extremely difficult to handle. Instead of attempting to derive Jacobi's condition rigorously by means of this equation, we shall content ourselves with using a simpler and less rigorous method.†

According to the last theorem the value of I_1 along the composite arc $E_{03}+N_{32}+E_{21}$ is always the same as its value along E_{02}. Since N_{32} is not an extremal, it can be replaced by an arc C_{34} giving I_1 a larger value, and hence $I_1(E_{01})$ cannot be a minimum.‡ As a further necessary condition we must, therefore, demand that there be no point 2 conjugate to 0 between 0 and 1 on a maximizing arc E_{01} which is an extremal, with a condition analogous to the condition $\partial^2 F_1/\partial y'\partial y' \neq 0$ everywhere on it.

IV. Sufficient conditions for simultaneous maxima

9. Relative strong and weak maxima. By definition an extremal curve E_{01} furnishes a *strong relative maximum* for an integral I_1 when u_3 is not allowed to vary, if there exist a positive number ϵ such that the integral

---

* Bliss, loc. cit., p. 163.

† Bliss, loc. cit., p. 141.

‡ To prove this we need to know that the functional-differential equations (9) defining an extremal have a unique solution at an arbitrarily selected point and direction.

$$I_1 = \int_{x_0}^{x_1} F_1(u_1, u_1', \cdots, u_4, u_4', x)dx$$

is greater than the integral

$$I_1(w) = \int_{x_0}^{x_1} F_1[u_1 + w_1(x), u_1' + w_1'(x), \cdots, u_4 + w_4(x), u_4' + w_4'(x), x]dx$$

for all possible forms of the functions w_σ, $\sigma = 1, \cdots, 4$, of class (I) in the interval $x_0 \leqq x \leqq x_1$, and satisfying the conditions*

$$w_\sigma(x_0) = 0\,;\ \ | w_\sigma(x) | < \epsilon\,;\ x_0 \leqq x \leqq x_1. \tag{18}$$

When in addition to the conditions above the functions $w_\sigma(x)$ satisfy $| w_\sigma'(x) | < \epsilon$ for $x_0 \leqq x \leqq x_1$, an extremal curve (u_1, u_2, u_3, u_4, x) furnishes a *weak relative maximum.*†

10. Sufficient conditions for a maximum. By means of the definition of an extremal field of §6 and the definitions of strong and weak relative maxima of §9 we are now in a position to write sufficient conditions for both strong and weak relative maxima.

THEOREM 8. *If E_{01} is an extremal arc and if the conditions* (14) *and* (15) *without the equality sign are satisfied at every element $(u_1, u_1', \cdots, u_4, u_4', x)$ in a neighborhood R', contained in R, of the corresponding elements of E_{01} for every admissible set $(u_1, U_1', \cdots, u_4, U_4', x)$ such that in* (14) *the expressions $(U_1' - u_1')$ and $(U_2' - u_2')$ are not both zero and yet $U_3' - u_3' \equiv 0$, and, furthermore, in* (15) *$(U_3' - u_3')$ is not zero and yet $U_1' - u_1' \equiv 0$ and $U_2' - u_2' \equiv 0$, and if, finally, there is no point* 2 *conjugate to* 0 *between* 0 *and* 1 *on E_{01}, then $I_1(E_{01})$ is a strong relative maximum when u_3 is not allowed to vary, and $I_2(E_{01})$ is a strong relative maximum when u_1 and u_2 are not allowed to vary.*

The conditions for a weak relative maximum do not require that the Weierstrass conditions be satisfied.

THEOREM 9. *If E_{01} is an extremal arc and if the Legendre conditions* (16) *and* (17) *without the equality sign are satisfied at every set of values $(u_1, u_1', \cdots, u_4, u_4', x)$ on this arc, and if there is no point* 2 *conjugate to* 0 *between* 0 *and* 1 *on E_{01}, then $I_1(E_{01})$ is a weak relative maximum when u_3 is not allowed to vary, and $I_2(E_{01})$ is a weak relative maximum when u_1 and u_2 are not allowed to vary.*

* If $w(x)$ is a continuous function admitting a continuous derivative in $x_0 \leqq x \leqq x_1$, we shall say that it belongs to the class (I) in the interval $(x_0 x_1)$. See E. Goursat, *Cours d'Analyse Mathématique*, vol. 3, p. 547.

† These are the classical definitions given by Goursat, loc. cit., pp. 612–613.

Although the above sufficient conditions apply strictly to the generalized Lagrange problem, by a slight modification they can be made to apply to the classical problem where only one integral I_1 is considered. Thus, allowing u_3 to vary in F_1 requires that the subscript k in (14) and (15) take on the values 1, 2, 3 instead of 1 and 2 only. The arguments of F_1 and G in all of the relations must of course be changed so that U_3' is given a proper place. In as much as this change will be obvious to the careful reader no attempt to write the corresponding conditions for the classical problem will be made here.

*The sufficiency theorems for the general problem when one end point is variable differ from those just given in that the transversality conditions must be adjoined.**

V. Integral equation treatment of the problem of Lagrange for more than one differential equation

11. Equations of variation. It is readily seen that the analysis of the preceding chapter applies to the problem of Lagrange for one differential equation. By introducing the theory of Volterra integral equations† this analysis can be modified to apply to the Lagrange problem for more than one differential relation and to the more general problem for which the differential relation is replaced by a functional relation of the type referred to in the introduction. Since the method employed in solving the problem for two differential equations is perfectly general, we need only discuss this case.

Our problem is to determine, through two fixed points 0 and 1 in the hyperspace (u_1, u_2, u_3, x), a curve E_{01} which satisfies two differential equations $G_k(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)=0$, $k=1, 2$, and which furnishes a maximum for an integral

$$I = \int_{x_0}^{x_1} F(u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x)dx.$$

We assume the G_k to be functionally independent, i.e.

$$\Delta G = \begin{vmatrix} \partial G_1/\partial u_2' & \partial G_1/\partial u_3' \\ \partial G_2/\partial u_2' & \partial G_2/\partial u_3' \end{vmatrix} \neq 0$$

in the region under consideration, and to possess continuous second-order partial derivatives with respect to $u_1, u_1', u_2, u_2', u_3, u_3', u_4, u_4', x$. Although

* Bliss, loc. cit., pp. 169–170.

† For the theory of Volterra integral equations see V. Volterra, *Leçons sur les Equations Intégrales.*

we have chosen both ends fixed, it is not necessary to make this assumption, as will appear presently. Let us first consider the problem stated above for both end points fixed.

Let the maximizing curve E_{01}, if such a curve exist, be the one defined by the equations

$$u_p = z_p(x) \qquad (p = 1,2),$$

$$u_3 = y(x) \qquad (x_0 \leqq x \leqq x_1)$$

and write

$$u_3 = y + \theta(x,a)\ ;\ u_p = z_p + f_p(x,a) \qquad (p = 1,2),$$

where θ and f_p are functions continuous with their second derivatives with respect to x and a, and which vanish when a vanishes. This notation for the u_i, $i=1, 2, 3$, is used to indicate that u_3 is to be regarded as the function whose variation is independent. We assume the variations of u_1 and u_2 to be determined by the end values $u_1(x_0)=u_{10}$, $u_2(x_0)=u_{20}$ and the differential equations of total variation

$$(19) \quad (\partial G_k/\partial y)\delta\theta + (\partial G_k/\partial y')\delta\theta' + \sum_{p=1}^{2}\left[(\partial G_k/\partial z_p)\delta f_p + (\partial G_k/\partial z_p')\delta f_p'\right] = 0 \qquad (k = 1,2)$$

obtained by substituting the values of u_i given above in the differential equations $G_k=0$, differentiating with respect to a and then setting $a=0$. As we showed in §3 this is consistent with the assumption that 0 is a fixed end point.

If these same values of u_i be substituted in F, the integral I becomes a function of the parameter a and yields on differentiation with respect to this parameter

$$(\partial I/\partial a)\delta a = \int_{x_0}^{x_1}\left[(\partial F/\partial y)\delta\theta + (\partial F/\partial y')\delta\theta' + (\partial F/\partial z_r)\delta f_r + (\partial F/\partial z_r')\delta f_r'\right]dx,$$

where r is an umbral index with range 1, 2 corresponding to p with range 1, 2.

12. Dependent variations by the theory of Volterra integral equations. In the classical treatment of this problem, Lagrange multipliers are introduced at this stage, but they can be advantageously avoided by integrating equation (19) with respect to x between the limits x_0 and x, where $x_0 \leqq x \leqq x_1$. If we perform this integration, replace x under the integral sign by s, and then perform an integration by parts on the terms which involve the primed variations, we obtain

$$(20)\qquad (\partial G_k/\partial z_r')\delta f_r + (\partial G_k/\partial y')\delta\theta + \int_{x_0}^{x}\left[\partial G_k/\partial y - \frac{d}{ds}(\partial G_k/\partial y')\right]\delta\theta ds + \int_{x_0}^{x}\left[\partial G_k/\partial z_r - \frac{d}{ds}(\partial G_k/\partial z_r')\right]\delta f_r\, ds = 0,$$

for, since $\delta\theta$ vanishes at x_0, the δf_r must also vanish if the determinant ΔG is not zero in $x_0 \leqq x \leqq x_1$. We take r umbral as before.

The variations δf_r are then determined by the system of Volterra integral equations

$$(21)\qquad \delta f_r(x) = \phi_r(x) + \int_{x_0}^{x} K_{rp}(x,s)\delta f_p(s)ds \qquad (r = 1,2),$$

where by definition

$$\phi_r(x) = A_{hr}(x)\left[(\partial G_h/\partial y')\delta\theta + \int_{x_0}^{x}\left[\partial G_h/\partial y - \frac{d}{ds}(\partial G_h/\partial y')\right]\delta\theta\, ds\right],$$

$$K_{rp}(x,s) = A_{hr}(x)\left[(\partial G_h(s)/\partial z_p) - \frac{d}{ds}(\partial G_h(s)/\partial z_p')\right],$$

where h and p are umbral indices having ranges 1, 2, and A_{hr} is the cofactor of the corresponding element of ΔG divided by $-\Delta G$.

These integral equations form a Volterra system of the second type for the determination of the δf_r, uniquely, if the kernels $K_{rp}(x, s)$ are finite and integrable in the interval $x_0 \leqq s \leqq x \leqq x_1$.*

If $\Delta G \neq 0$ on the range $x_0 \leqq s \leqq x \leqq x_1$, the $K_{rp}(x, s)$ will be finite and integrable on the range because of the continuity requirements on the G_k. The unique solution of the system is, therefore,

$$(22)\qquad \delta f_r(x) = \phi_r(x) + \int_{x_0}^{x} S_{rp}(x,s)\phi_p(s)ds,$$

where p is umbral with range 1, 2 and $S_{rp}(x, s)$ is the resolvent kernel of $K_{rp}(x, s)$ defined by the equations

$$K_{rp}^{1}(x,s) = -K_{rp}(x,s),$$

$$K_{rp}^{i}(x,s) = \int_{s}^{x} K_{rh}^{\sigma}(x,t)K_{hp}^{i-\sigma}(t,s)dt,$$

$$S_{rp}(x,s) = \sum_{i=1}^{\infty} K_{rp}^{i}(x,s).$$

* Volterra, loc. cit., p. 71.

If we substitute in (22) the values of $\phi_r(x)$ and $\phi_p(s)$ as given by their definitions, then apply Dirichlet's formula to the iterated integral of the result, and then interchange the parameters of integration, we may write the variation of δf_r as

$$\delta f_r(x) = W_r(x)\delta\theta + \int_{x_0}^{x} V_r(x,s)\delta\theta\, ds, \tag{23}$$

where by definition

$$W_r(x) = A_{hr}(x)(\partial G_h/\partial y'),$$

$$V_r(x,s) = A_{hr}(x)\left[\partial G_h/\partial y - \frac{d}{ds}(\partial G_h/\partial y')\right] + A_{hp}(s)S_{rp}(x,s)(\partial G_h/\partial y')$$
$$+ A_{hp}(s)\left[\partial G_h/\partial y - \frac{d}{ds}(\partial G_h/\partial y')\right]\int_{s}^{x} S_{rp}(x,t)dt,$$

and where h and p are umbral indices with range 1, 2.

By differentiation with respect to x we obtain

$$\delta f_r'(x) = \frac{d}{dx}[W_r\delta\theta] + V_r(x,x)\delta\theta + \int_{x_0}^{x}[\partial V_r(x,s)/\partial x]\delta\theta\, ds. \tag{24}$$

13. Eulerian equations. A substitution of $\delta f_r(x)$ and $\delta f_r'(x)$ in the first variation of I followed by an application of Dirichlet's formula as before yields

$$(\partial I/\partial a)\delta a = \int_{x_0}^{x_1}[\partial F/\partial y + (\partial F/\partial z_r)W_r(x) + V_r(x,x)(\partial F/\partial z_r') + T(x)]\delta\theta\, dx$$
$$+ \int_{x_0}^{x_1}[(\partial F/\partial y')\delta\theta' + (\partial F/\partial z_r')d(W_r\delta\theta)/dx]dx,$$

where

$$T(x) = \int_{x_0}^{x_1}[(\partial F/\partial z_r)V_r(s,x) + (\partial F/\partial z_r')(\partial V_r(s,x)/\partial s)]ds.$$

An integration by parts performed on the primed terms yields

$$\delta I = \int_{x_0}^{x_1}\left[\partial F/\partial y + (\partial F/\partial z_r)W_r + V_r(x,x)(\partial F/\partial z_r') + T(x) - \frac{d}{dx}(\partial F/\partial y')\right.$$
$$\left. - W_r\frac{d}{dx}(\partial F/\partial z_r')\right]\delta\theta\, dx.$$

In order that δI vanish for all $\delta\theta$ it is necessary that the coefficient of $\delta\theta$ in the above integral vanish and hence

$$(25)\quad \partial F/\partial y + (\partial F/\partial z_r)W_r + V_r(x,x)\,(\partial F/\partial z_r') + T(x) - \frac{d}{dx}(\partial F/\partial y') - W_r\frac{d}{dx}(\partial F/\partial z_r') = 0,$$

where r is an umbral index with range 1, 2.

It is well to note that the function $T(x)$ is an integral involving the resolvent kernel of the system of Volterra integral equations defining the variations. If some variable other than u_3 had been chosen to be independent, a different set of conditions of the type (25) would result, but presumably the new set would be equivalent to (25).

If one or both end points were variable, the problem could still be treated by the methods of this paragraph. For this case equation (20) would contain terms in $\delta\theta(x_0)$, $\delta\theta(x_1)$, $\delta f_r(x_0)$, $\delta f_r(x_1)$. These terms could be carried all the way through the analysis and would yield transversality conditions in a new form. This interesting problem will not be attacked in the present paper.

VI. Further generalizations

14. **Problem for functional relations.** A special problem in which a linear integral equation replaces the first-order differential equation $G=0$ has already been considered.* We desire now to consider the more general problem of determining a curve E_{01} of the space (u_1, u_2, u_3, x) satisfying functional relations

$$(26)\quad \begin{aligned} &G_k(u_1,u_1',u_2,u_2',u_3,u_3',x) \\ &= \int_{x_0}^{x} P_k(u_1,u_1',u_2,u_2',u_3,u_3',x,s)ds \qquad (k=1,2), \end{aligned}$$

such that an integral

$$I = \int_{x_0}^{x_1} F(u_1,u_1',u_2,u_2',u_3,u_3',x)dx$$

is a maximum. We may suppose the end parameter x_0 and the corresponding end values $u_i(x_0)$, $i=1, 2, 3$, to be fixed, although this is not necessary. Let us, for the sake of brevity, also suppose x_1 and the corresponding end values of the u_i to be fixed.

If the u_i are replaced by functions satisfying the same conditions as the corresponding functions of §11, the functional equations (26) become relations involving the parameter a and yield by parametric differentiation

---

* *A mathematical theory of competition*, loc. cit., p. 173.

$$
\begin{aligned}
(\partial G_k/\partial y)\delta\theta + (\partial G_k/\partial y')\delta\theta' + (\partial G_k/\partial z_r)\delta f_r + (\partial G_k/\partial z_r')\delta f_r'& \\
= \int_{x_0}^{x} [(\partial P_k/\partial y)\delta\theta + (\partial P_k/\partial y')\delta\theta'& \\
+ (\partial P_k/\partial z_r)\delta f_r + (\partial P_k/\partial z_r')\delta f_r']ds& \qquad (k = 1,2),
\end{aligned}
$$

where r is umbral with range 1, 2.

An integration with respect to x followed by an integration by parts on the primed variations yields

$$
\begin{aligned}
(\partial G_k/\partial y')\delta\theta &+ \int_{x_0}^{x}\left[\partial G_k/\partial y - \frac{d}{ds}(\partial G_k/\partial y')\right]\delta\theta\, ds + (\partial G_k/\partial z_r')\delta f_r \\
&+ \int_{x_0}^{x}\left[\partial G_k/\partial z_r - \frac{d}{ds}(\partial G_k/\partial z_r')\right]\delta f_r\, ds \\
&= \int_{x_0}^{x} [(\partial P_k/\partial y')\delta\theta + (\partial P_k/\partial z_r')\delta f_r]ds \\
&\quad + \int_{x_0}^{x} ds \int_{x_0}^{s}\left[\left(\partial P_k/\partial y - \frac{d}{dt}(\partial P_k/\partial y')\right)\delta\theta \right.\\
&\quad \left. + \left(\partial P_k/\partial z_r - \frac{d}{dt}(\partial P_k/\partial z_r')\right)\delta f_r\right]dt,
\end{aligned}
$$

where for convenience in notation the parameter of integration x has been changed to s.

If we apply Dirichlet's formula to the iterated integral, we can write this expression as

$$
\begin{aligned}
(\partial G_k/\partial z_r')\delta f_r = -\, (\partial G_k/\partial y')\delta\theta &- \int_{x_0}^{x}\left[\partial G_k/\partial y - \frac{d}{ds}(\partial G_k/\partial y') - \partial P_k/\partial y' \right.\\
&\left. - (x - s)\left(\partial P_k/\partial y - \frac{d}{ds}(\partial P_k/\partial y')\right)\right]\delta\theta\, ds \\
- \int_{x_0}^{x}\left[\partial G_k/\partial z_r \right.&- \frac{d}{ds}(\partial G_k/\partial z_r') - \partial P_k/\partial z_r' \\
&\left. - (x - s)\left(\partial P_k/\partial z_r - \frac{d}{ds}(\partial P_k/\partial z_r')\right)\right]\delta f_r\, ds.
\end{aligned}
$$

As far as the variations of δf_r and $\delta\theta$ are concerned this expression is of the same form as (21), if $\partial G_k/\partial z_r'$ is not zero on the range $x_0 \leqq s \leqq x \leqq x_1$. The analysis of the preceding section, therefore, applies from this point.

15. **Further extensions.** The extension to the case of more than one independent variable is obtained by placing a subscript on y in the above equations and regarding this subscript as an umbral index of the proper range.

The problem of simultaneous maxima for more than one differential or integral relation can be treated by this same method, since, if there are two integrals and two independent variables u_3 and u_4, equation (25), with the proper arguments for F_1, F_2 and G_k substituted, is a necessary condition that a curve E_{01} in the space (u_1, u_2, u_3, u_4, x) satisfy differential equations $G_k(u_1, u_1', \cdots, u_4, u_4', x)=0$, $k=1, 2$, and make an integral

$$I = \int_x^{x_1} F_1(u_1, u_1', \cdots, u_4, u_4', x)dx$$

a maximum when u_4 is not allowed to vary.

A discussion of the corresponding problems for variable end points should lead to analogues of the Weierstrass and Legendre necessary conditions and to sufficient conditions for strong and weak relative maxima.

The assumption that the admissible arcs have continuously turning tangents is by no means necessary. If we make assumptions on admissible arcs similar to those of Part II, we can obtain the analogues of the more general Euler equations (7) and (8), by integrating by parts the terms in $\delta\theta$ instead of those in $\delta\theta'$. The analogues of the Weierstrass-Erdmann corner conditions* follow readily from the Euler equations in the form (7) and (8).

* Bliss, loc. cit., p. 143, and O. Bolza, *Vorlesungen über Variationsrechnung*, 1909, p. 366.

RICE INSTITUTE,
HOUSTON, TEXAS